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Berglund's Diffusion Constant Model

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1 Introduction

I worked for a long time in a variation of this model, for various reasons it did not get published but I wanted to share part of the derivation with the community - as it is not obvious in the paper - especially if you don't do calculus every day!!. I hope this will be useful for someone else.

2 Model

A particle moves on one dimension with time intervals of size τ on positions $\{x_1, x_2, \ldots, x_N\}$.

The Brownian motion model is governed by the following equation

$$x_n = x_{n-1} + \epsilon_n$$

with $\epsilon_n \sim \mathcal{N}(0, \sigma_\Delta^2)$

Where $\sigma_\Delta^2 = 2D\tau$ with D the diffusion constant.

We also have measurement noise σ_c (also called localization noise) which is usually modeled as having observations $\{y_1, y_2, \ldots, y_N\}$ consisting of the true position and some measurement error:

$$y_n = x_n + \phi_n$$

where $\phi_n \sim \mathcal{N}(0, \sigma_c^2)$.

However, the method suggested by Berglund (Berglund (2010), see also Michalet and Berglund (2012)) adds the fact that the particle is being imaged by a camera and subject to the shutter being opened in a given interval of time, giving rise to motion blur. Therefore he defines a shutter function *s(t)* that integrates to unity over the time interval τ and defines the observed position on frame k, Y_k, as the true position X averaged by the shutter function plus the measurement noise ϕ_k,

$$y_k = \int_{(k-1)\tau}^{k\tau} s(t - (k-1)\tau) X(t) dt + \phi_k$$

Where $\phi_k \sim \mathcal{N}(0, \sigma_c^2)$.

This leads to the following multivariate normal distribution for the displacements $\Delta_k = y_{k+1} - y_k$:

$$\langle \Delta_i \Delta_j \rangle = \begin{cases} 2D\tau - 2(2DR\tau - \sigma_c^2) = \sigma - 2R\sigma - 2\sigma_c^2 \text{ if } i = j \\ 2DR\tau - \sigma_c^2 = \sigma R - \sigma_c^2 \text{ if } |i-j| = 1 \\ 0 \text{ otherwise} \end{cases}$$

where, if $S(t) = \int_0^t s(u)du$,

$$R = \frac{1}{\tau}\left(\int_0^\tau s(t')t'dt' - \int_0^\tau \int_0^\tau s(t)s(t')min(t,t')dtdt'\right) = \int_0^\tau S(t)(1 - S(t)dt$$

R would be a *motion blur coefficient* with $0 \leq R \leq 1/4$.

(see Berglund (2010) and the supplement for the derivation)

Therefore we have the likelihood as

$$p(y_1, \ldots, y_p|\sigma, \sigma_c, R) = p(\Delta_1, \ldots, \Delta_{p-1}) = \mathcal{N}_p(0, \sum)$$

Where $\sum$ is as defined above.

Finally, we want to study the case where we know the sequence of hidden states $S = \{s_1, s_2, \ldots, s_N\}$, where $s_i \in \{1, \ldots, K\}$ with N the number of diffusion states and, instead of a single σ_Δ, we have $\{\sigma_1, \sigma_2, \ldots, \sigma_K\}$ as the actual diffusion states. Then we will set the joint distribution including the hidden states to be the product of the separate joint distributions for each state:

$$\begin{aligned} p(Y, \sigma_1, \ldots, \sigma_K, \sigma_c) &= p(y_1, \ldots, y_N, s_1, \ldots, s_N, \sigma_1, \ldots, \sigma_K, \sigma_c) \\ &= \left[\prod_{k=1}^{K} p(y_{k,1}, \ldots, y_{k,n_k}|\sigma_k, \sigma_c)\right] p(\sigma_1, \ldots, \sigma_k) \end{aligned}$$

Supplementary Material

The method suggested by Berglund and Michalet (Berglund (2010); Michalet and Berglund (2012)) add the fact that the particle is being imaged by a camera and subject to the shutter being opened in a given interval of time, giving rise to motion blur. Therefore we define a shutter function *s(t)* that integrates to unity over the time interval τ and define the observed position on frame k, Y_k, as the true position X averaged by the shutter function plus the localization noise ϕ_k,

$$Y_k = \int_{(k-1)\tau}^{k\tau} s(t - (k-1)\tau)X(t)dt + \phi_k$$

Where $\phi_k \sim \mathcal{N}(0, \sigma_c^2)$.

We also assume that the particle follows Brownian motion so,

$$\langle \epsilon_k^2 \rangle = \langle (X((k+1)\tau) - X(k\tau))^2 \rangle = 2D\tau$$

For the derivation below we will use the following result:

$$\begin{aligned}
\langle X_{true}(t')X_{true}(t'')\rangle &= \langle [X_{true}(t'-1) + \epsilon_{t'}][X_{true}(t''-1) + \epsilon_{t''}]\rangle \\
&= \langle [X_{true}(t'-2) + \epsilon_{t'-1} + \epsilon_{t'}][X_{true}(t''-2) + \epsilon_{t''-1} + \epsilon_{t''}]\rangle \\
&= \dots \\
&= \langle [X_{true}(0) + \sum_{i=1}^{t'} \epsilon_i][X_{true}(0) + \sum_{j=1}^{t''} \epsilon_j]\rangle \\
&= \langle X_{true}^2(0) + X_{true}(0)\sum_{j=1}^{t'} \epsilon_j + X_{true}(0)\sum_{i=1}^{t''} \epsilon_i + \sum_{i=1}^{t''} \epsilon_i \sum_{j=1}^{t''} \epsilon_j \rangle \\
&= \langle X_{true}^2(0)\rangle + \langle X_{true}(0)\sum_{j=1}^{t'} \epsilon_j\rangle + \langle X_{true}(0)\sum_{i=1}^{t''} \epsilon_i\rangle + \langle \sum_{i=1}^{t''} \epsilon_i \sum_{j=1}^{t''} \epsilon_j \rangle \\
&= X_{true}^2(0) + \langle \sum_{i=1}^{t''} \epsilon_i \sum_{j=1}^{t''} \epsilon_j \rangle \\
&= X_{true}^2(0) + \langle \sum_{i=1}^{min(t',t'')} \epsilon_i^2 + \sum_{j=min(t',t'')+1}^{max(t',t'')} \epsilon_j \rangle \\
&= X_{true}^2(0) + \sum_{i=1}^{min(t',t'')} \langle \epsilon_i^2 \rangle + \sum_{j=min(t',t'')+1}^{max(t',t'')} \langle \epsilon_j \rangle \\
&= X_{true}^2(0) + min(t',t'')2D
\end{aligned} \tag{2.1}$$

We now want to find the distribution for the measured displacements $\Delta_k = Y_{k+1} - Y_k$. First we see that, using the substitution $u = t - (k-1)\tau$:

$$Y_k = \int_{(k-1)\tau}^{k\tau} s(t-(k-1)\tau)X(t)dt + \phi_k \quad = \int_0^{\tau} s(u+(k-1)\tau)X(u)du$$

And we have:

$$\begin{aligned}
\langle \Delta_k \Delta_{k-1}\rangle &= \langle (Y_{k+1} - Y_k)(Y_k - Y_{k-1})\rangle \\
&= \langle (Y_{k+1}Y_k) - (Y_{k+1}Y_{k-1}) - (Y_kY_k) + (Y_kY_{k-1})\rangle \\
&= \langle Y_{k+1}Y_k\rangle - \langle Y_{k+1}Y_{k-1}\rangle - \langle Y_kY_k\rangle + \langle Y_kY_{k-1}\rangle
\end{aligned}$$

Using (2.1) we can determine $\langle Y_{k+1}Y_k\rangle$, $\langle Y_kY_k\rangle$ and $\langle Y_{k+1}Y_{k-1}\rangle$ as follows

$$
\begin{aligned}
\langle Y_{k+1}Y_k\rangle &= \langle(\int_0^\tau s(t)X(t+k\tau)dt+\phi_{k+1})(\int_0^\tau s(t')X(t'+(k-1)\tau)dt'+\phi_k)\rangle \\
&= \langle\int_0^\tau s(t)X(t+k\tau)dt\int_0^\tau s(t')X(t'+(k-1)\tau)dt'\rangle+\langle\phi_k\int_0^\tau s(t)X(t+k\tau)dt\rangle+\langle\phi_{k+1}\int_0^\tau s(t')X(t'+(k-1)\tau)dt'\rangle+\langle\phi_{k+1}\phi_k\rangle \\
&= \langle\int_0^\tau\int_0^\tau s(t)s(t')X(t+k\tau)X(t'+(k-1)\tau)dtdt'\rangle \\
&= \int_0^\tau\int_0^\tau s(t)s(t')\langle X(t+k\tau)X(t'+(k-1)\tau)\rangle dtdt' \\
&= \int_0^\tau\int_0^\tau s(t)s(t')[X(0)^2+2D(t'+(k-1))\tau)]dtdt' \\
&= X(0)^2\int_0^\tau\int_0^\tau s(t)s(t')dtdt'+2D\int_0^\tau\int_0^\tau s(t)s(t')tdtdt'+2D(k-1)\tau\int_0^\tau\int_0^\tau s(t)s(t')dtdt' \\
&= X(0)^2+2D\int_0^\tau s(t)tdt+2D(k-1)\tau
\end{aligned}
\tag{2.2}
$$

and,

$$
\begin{aligned}
\langle Y_kY_k\rangle &= \langle(\int_0^\tau s(t)X(t+(k-1)\tau)dt+\phi_k)(\int_0^\tau s(t')X(t'+(k-1)\tau)dt'+\phi_k)\rangle \\
&= \langle\int_0^\tau s(t)X(t+(k-1)\tau)dt\int_0^\tau s(t')X(t'+(k-1)\tau)dt'\rangle+\langle 2\phi_k\int_0^\tau s(t')X(t'+(k-1)\tau)dt'\rangle+\langle\phi_k\phi_k\rangle \\
&= \langle\int_0^\tau\int_0^\tau s(t)s(t')X(t+(k-1)\tau)X(t'+(k-1)\tau)dtdt'\rangle+\sigma^2 \\
&= \int_0^\tau\int_0^\tau s(t)s(t')\langle X(t+(k-1)\tau)X(t'+(k-1)\tau)\rangle dtdt'+\sigma^2 \\
&= \int_0^\tau\int_0^\tau s(t)s(t')[X(0)^2+2D(min(t,t')+(k-1)\tau)]dtdt'+\sigma^2 \\
&= X(0)^2+2D\int_0^\tau\int_0^\tau s(t)s(t')min(t,t')dtdt'+2D(k-1)\tau\int_0^\tau\int_0^\tau s(t)s(t')dtdt'+\sigma^2 \\
&= X(0)^2+2D\int_0^\tau\int_0^\tau s(t)s(t')min(t,t')dtdt'+2D(k-1)\tau+\sigma^2
\end{aligned}
\tag{2.3}
$$

And

$$
\begin{aligned}
\langle Y_{k+1}Y_{k-1}\rangle &= \langle(\int_0^\tau s(t)X(t+k\tau)dt + \phi_{k+1})(\int_0^\tau s(t')X(t'+(k-2)\tau)dt' + \phi_{k-1})\rangle \\
&= \langle\int_0^\tau\int_0^\tau s(t)s(t')X(t+k\tau)X(t'+(k-2)\tau)dtdt'\rangle \\
&= \int_0^\tau\int_0^\tau s(t)s(t')\langle X(t+k\tau)X(t'+(k-2)\tau)\rangle dtdt' \\
&= \int_0^\tau\int_0^\tau s(t)s(t')[X(0)^2 + 2D(t'+(k-2)\tau)]dtdt' \\
&= X(0)^2 + 2D\int_0^\tau s(t')t'dt' + 2D(k-2)\tau
\end{aligned} \tag{2.4}
$$

Using (2.2), (2.3) and (2.4) together:

$$
\begin{aligned}
\langle\Delta_k\Delta_{k-1}\rangle &= \langle Y_{k+1}Y_k\rangle - \langle Y_{k+1}Y_{k-1}\rangle - \langle Y_kY_k\rangle + \langle Y_kY_{k-1}\rangle \\
&= X(0)^2 + 2D\int_0^\tau s(t')t'dt' + 2D(k-1)\tau \\
&\quad - (X(0)^2 + 2D\int_0^\tau s(t')t'dt' + 2D(k-2)\tau) \\
&\quad - (X(0)^2 + 2D\int_0^\tau\int_0^\tau s(t)s(t')min(t,t')dtdt' + 2D(k-1)\tau) + \sigma^2) \\
&\quad + X(0)^2 + 2D\int_0^\tau s(t')t'dt' + 2D(k-2)\tau \\
&= 2D\int_0^\tau s(t')t'dt - 2D\int_0^\tau\int_0^\tau s(t)s(t')min(t,t')dtdt' - \sigma^2 \\
&= 2D\left(\int_0^\tau s(t')t'dt - \int_0^\tau\int_0^\tau s(t)s(t')min(t,t')dtdt'\right) - \sigma^2
\end{aligned}
$$

Analogously,

$$\begin{aligned}
\langle \Delta_k \Delta_k \rangle &= \langle (Y_{k+1} - Y_k)(Y_{k+1} - Y_k \rangle \\
&= \langle Y_{k+1} Y_{k+1} \rangle - 2 \langle Y_{k+1} Y_k \rangle - \langle Y_k Y_k \rangle \\
&= (X(0)^2 + 2D \int_0^\tau \int_0^\tau s(t)s(t') min(t,t') dt dt' + 2Dk\tau) + \sigma^2) \\
&\quad - 2 \left(X(0)^2 + 2D \int_0^\tau s(t')t' dt' + 2D(k-1)\tau \right) \\
&\quad (X(0)^2 + 2D \int_0^\tau \int_0^\tau s(t)s(t') min(t,t') dt dt' + 2D(k-1)\tau) + \sigma^2) \\
&= 4D \int_0^\tau \int_0^\tau s(t)s(t') min(t,t') dt dt' - 4D \int_0^\tau s(t')t' dt' + 2D\tau + 2\sigma^2 \\
&= 2D\tau - 2 \left[2D \left(\int_0^\tau s(t')t' dt' - \int_0^\tau \int_0^\tau s(t)s(t') min(t,t') dt dt' \right) - \sigma^2 \right]
\end{aligned}$$

We also have that for the case $|k - j| > 1$, assuming without loss of generality that k¿j,

$$\begin{aligned}
\langle \Delta_k \Delta_j \rangle &= \langle (Y_{k+1} - Y_k)(Y_{j+1} - Y_j \rangle \\
&= \langle Y_{k+1} Y_{j+1} \rangle - \langle Y_{k+1} Y_j \rangle - \langle Y_k Y_{j+1} \rangle + \langle Y_k Y_j \rangle \\
&= 2Dj\tau - 2D(j-1)\tau - 2Dj\tau + 2D(j-1)\tau \\
&= 0
\end{aligned}$$

Hence if we define $R = \frac{1}{\tau} \left(\int_0^\tau s(t')t' dt' - \int_0^\tau \int_0^\tau s(t)s(t') min(t,t') dt dt' \right)$ we see that the displacements have a multivariate normal distribution centered at zero and with a co-variance matrix as follows:

$$\begin{aligned}
\langle \Delta_k \Delta_{k-1} \rangle &= 2D\tau R - \sigma^2 \\
\langle \Delta_k \Delta_k \rangle &= 2D\tau - 2\left(2D\tau R - \sigma^2\right) \\
\langle \Delta_i \Delta_j \rangle &= 0 \text{ if —i-j—¿1}
\end{aligned}$$

This implies correlation between consecutive displacements.

In addition, if we define $S(t) = \int_0^t s(u) du$ we have:

$$R = \frac{1}{\tau} \left(\int_0^\tau s(t')t' dt' - \int_0^\tau \int_0^\tau s(t)s(t') min(t,t') dt dt' \right) = \int_0^\tau S(t)(1 - S(t) dt$$

This can be seen by the following:

First, using integration by parts:

$$\begin{aligned}\int_0^\tau t's(t')dt' &= t'(S(t)-1)|_0^\tau - \int_0^\tau (S(t)-1)dt \\ &= 0 - \int_0^\tau (S(t)-1)dt \\ &= \int_0^\tau (1-S(t))dt\end{aligned} \tag{2.5}$$

Then we also have, using an index function where

$$\mathbb{1}_{\{\ldots\}} = \begin{cases} 1, & \text{if } \ldots \text{ is true,} \\ 0, & \text{otherwise} \end{cases}$$

So for $t, t' \in [0, \tau]$

$$min\{t, t'\} = \int_0^\tau \mathbb{1}_{\{u\le t, u\le t'\}} du \quad = \int_0^\tau \mathbb{1}_{\{u\le t\}} \mathbb{1}_{\{u\le t'\}} du$$

And

$$\begin{aligned}\int_0^\tau \int_0^\tau s(t)s(t')min(t,t')dtdt' &= \int_0^\tau \int_0^\tau s(t)s(t') \int_0^\tau \mathbb{1}_{\{u\le t\}} \mathbb{1}_{\{u\le t'\}} dudtdt' \\ &= \int_0^\tau \int_0^\tau s(t)\mathbb{1}_{\{u\le t\}} dt \int_0^\tau s(t')\mathbb{1}_{\{u\le t'\}} dt' du \\ &= \int_0^\tau \int_u^t s(t)dt \int_u^{t'} s(t')dt' du \\ &= \int_0^\tau (1-S(u))^2 du\end{aligned} \tag{2.6}$$

Now, combining (2.5) and (2.6)

$$\begin{aligned}R = \frac{1}{\tau}\left(\int_0^\tau s(t')t'dt' - \int_0^\tau \int_0^\tau s(t)s(t')min(t,t')dtdt'\right) &= \frac{1}{\tau}\int_0^\tau (1-S(u))du - \int_0^\tau (1-S(u))^2 du \\ &= \frac{1}{\tau}\int_0^\tau (1-S(u))\left(1-(1-S(u))\right) du \\ &= \frac{1}{\tau}\int_0^\tau (1-S(u))S(u)du\end{aligned}$$

Derivation of Berglund's Estimator. Derivation of Berglund's Estimator

References

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